

THE Q-TENSOR SQUARE OF FINITELY GENERATED NILPOTENT GROUPS, $q \geq 0$

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ABSTRACT. In the present paper the authors extend to the q -tensor square $G \otimes^q G$ of a group G , q a non-negative integer, some structural results due to R. D. Blyth, F. Fumagalli and M. Morigi concerning the non-abelian tensor square $G \otimes G$ ($q = 0$). The results are applied to the computation of $G \otimes^q G$ for finitely generated nilpotent groups G , specially for free nilpotent groups of finite rank. We also generalize to all $q \geq 0$ results of M. Bacon regarding an upper bound to the minimal number of generators of the non-abelian tensor square $G \otimes G$ when G is a n -generator nilpotent group of class 2. We end by computing the q -tensor squares of the free n -generator nilpotent group of class 2, $n \geq 2$, for all $q \geq 0$. This shows that the above mentioned upper bound is also achieved for these groups when $q > 1$.

1. INTRODUCTION

Let G and G^φ be groups, isomorphic via $\varphi : g \mapsto g^\varphi$ for all $g \in G$. Consider the group $\nu(G)$, introduced in [23] as

$$(1) \quad \nu(G) = \langle G \cup G^\varphi \mid [g, h^\varphi]^k = [g^k, (h^k)^\varphi] = [g, h^\varphi]^{k^\varphi}, \forall g, h, k \in G \rangle.$$

It is a well known fact (see [23]) that the subgroup $\Upsilon(G) = [G, G^\varphi]$ of $\nu(G)$ is isomorphic to the non-abelian tensor square $G \otimes G$, as defined by Brown and Loday in their seminal paper [8]. A modular version of the operator ν was considered in [10], where for any non-negative integer q the authors introduced and studied a group $\nu^q(G)$, which in turn is an extension of the so called q -tensor square of G , $G \otimes^q G$, first defined by Conduché and Rodriguez-Fernandez in [11] (see also [14], [7]). In order to describe the group $\nu^q(G)$, if $q \geq 1$ then let $\widehat{\mathcal{G}} = \{\widehat{k} \mid k \in G\}$ be a set of symbols, one for each element of G (for $q = 0$ we set $\widehat{\mathcal{G}} = \emptyset$, the empty set). Let $F(\widehat{\mathcal{G}})$ be the free group over $\widehat{\mathcal{G}}$ and $\nu(G) \star F(\widehat{\mathcal{G}})$ be the free product of $\nu(G)$ and $F(\widehat{\mathcal{G}})$. As G and G^φ are embedded into $\nu(G)$ we shall identify the elements of G (respectively of G^φ) with their respective images in $\nu(G) \star F(\widehat{\mathcal{G}})$. Let J denote the normal closure in

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$\nu(G) * F(\widehat{\mathcal{G}})$ of the following elements, for all $\widehat{k}, \widehat{k}_1 \in \widehat{\mathcal{G}}$ and $g, h \in G$:

$$(2) \quad g^{-1} \widehat{k} g (\widehat{k^g})^{-1};$$

$$(3) \quad (g^\varphi)^{-1} \widehat{k} g^\varphi (\widehat{k^g})^{-1};$$

$$(4) \quad (\widehat{k})^{-1} [g, h^\varphi] \widehat{k} [g^{k^q}, (h^{k^q})^\varphi]^{-1};$$

$$(5) \quad (\widehat{k})^{-1} \widehat{k k_1} (\widehat{k_1})^{-1} \left(\prod_{i=1}^{q-1} [k, (k_1^{-i})^\varphi]^{k^{q-1-i}} \right)^{-1};$$

$$(6) \quad [\widehat{k}, \widehat{k_1}] [k^q, (k_1^q)^\varphi]^{-1};$$

$$(7) \quad \widehat{[g, h]} [g, h^\varphi]^{-q}.$$

DEFINITION 1.1. The group $\nu^q(G)$ is defined to be the factor group

$$(8) \quad \nu^q(G) := (\nu(G) * F(\widehat{\mathcal{G}})) / J.$$

Note that for $q = 0$ the sets of relations (2) to (7) are empty; in this case we have $\nu^0(G) = \nu(G) * F(\widehat{\mathcal{G}}) / J \cong \nu(G)$.

Let R_1, \dots, R_6 be the sets of relations corresponding to (2), \dots , (7), respectively, and let R be their union, $R = \bigcup_{i=1}^6 R_i$. Therefore, $\nu^q(G)$ has the presentation:

$$\nu^q(G) = \langle G, G^\varphi, \widehat{\mathcal{G}} \mid R, [g, h^\varphi]^k [g^k, (h^k)^\varphi]^{-1}, [g, h^\varphi]^{k^\varphi} [g^k, (h^k)^\varphi]^{-1}, \forall g, h, k \in G \rangle.$$

There is an epimorphism $\rho : \eta^q(G) \twoheadrightarrow G, g \mapsto g, h^\varphi \mapsto h, \widehat{k} \mapsto k^q$. On the other hand the inclusion of G into $\nu(G)$ induces a homomorphism $\iota : G \rightarrow \nu^q(G)$. We have $g^{\iota\rho} = g$ and thus ι is injective. Similarly the inclusion of G^φ into $\nu(G)$ induces a monomorphism $j : G^\varphi \rightarrow \nu^q(G)$. Thus we shall identify the elements $g \in G$ and $g^\varphi \in G^\varphi$ with their respective images g^ι and $(g^\varphi)^j$ in $\nu^q(G)$.

Now let \mathfrak{G} denote the subgroup of $\nu^q(G)$ generated by the images of $\widehat{\mathcal{G}}$. By relations (4), \mathfrak{G} normalizes the subgroup $T = [G, G^\varphi]$ in $\nu^q(G)$ and hence $\Upsilon^q(G) = T\mathfrak{G} = [G, G^\varphi]\mathfrak{G}$ is a normal subgroup of $\nu^q(G)$. Thus we obtain $\nu^q(G) = G^\varphi \cdot (G \cdot \Upsilon^q(G))$, where the dots mean internal semidirect products. It should be noted that the actions of G and G^φ on $\Upsilon^q(G)$ are those induced by the defining relations of $\nu^q(G)$: for any elements $g, x \in G, h^\varphi, y^\varphi \in G^\varphi$ and $\widehat{k} \in \widehat{\mathcal{G}}$ we have $[g, h^\varphi]^x = [g^x, (h^x)^\varphi]$ and $(\widehat{k})^x = \widehat{(k^x)}$; similarly, $[g, h^\varphi]^{y^\varphi} = [g^y, (h^y)^\varphi]$ and $(\widehat{k})^{y^\varphi} = \widehat{(k^y)}$. In addition, for any $\tau \in \Upsilon^q(G)$, $(g\tau)^{y^\varphi} = g[g, y^\varphi]\tau^{y^\varphi} \in G\Upsilon^q(G)$.

By [10, Proposition 2.9] $\Upsilon^q(G)$ is isomorphic to the q -tensor square $G \otimes^q G$, for all $q \geq 0$. We then get a result (see [10, Corollary 2.11]) analogous to one due to Ellis in [14]: $\nu^q(G) \cong G \ltimes (G \ltimes (G \otimes^q G))$; this generalizes a similar result found in [23] for $q = 0$.

The commutator approach to $G \otimes G$ for the case $q = 0$, provided by the isomorphism between $G \otimes G$ and the subgroup $[G, G^\varphi]$ of $\nu(G)$ (see [23], and also [15]), has proven suitable to treat of non-abelian tensor products of groups, Schur multipliers and many other relevant invariants involving covering questions in groups; see for instance, references [15], [24], [20], [4], [12], [21] and the GAP Package ‘‘POLYCYCLIC’’ in [13].

The extension of the existing theory from $q = 0$ to all non-negative integers q , as addressed for instance in [10], broadens the scope of these connections, now in a *hat* (“power”) and *commutator* approach to the q -tensor square, $q \geq 0$.

In section 2 we extend to $G \otimes^q G$, $q \geq 0$, some structural results found in [5] and [24] concerning $G \otimes G$. In section 3 it is established an upper bound for the minimal number of generators of $G \otimes^q G$ when G is a finitely generated nilpotent group of class 2, thus generalizing a result of Bacon found in [2]. We end by computing the q -tensor square of the free nilpotent group of rank $n \geq 2$ and class 2, $\mathcal{N}_{n,2}$, $q \geq 0$; this will show, as in the case $q = 0$ (see [2, Theorem 3.2]), that the cited upper bound is also attained for these groups when $q > 1$, although in this case $\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2}$ is a non-abelian group.

Notation is fairly standard (see for instance [22]). If x and y are elements of a group G then we write y^x for the conjugate $x^{-1}yx$ and $[x, y]$ for the commutator $x^{-1}y^{-1}xy$. Our commutators are left normed: $[x, y, z] = [[x, y], z]$ for all $x, y, z \in G$, and so on, recursively, for commutators of higher weights. The order of x (resp. of G) is written $o(x)$ (resp. $|G|$). As usual, $\gamma_i(G)$ denotes the i^{th} term of the lower central series of G . For future reference we recall the well known Hall-Witt identity:

$$(9) \quad [x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1, \quad \forall x, y, z \in G.$$

In view of the isomorphism given by [10, Proposition 2.9], from now on we identify $G \otimes^q G$ with the subgroup $\Upsilon^q(G) = [G, G^q] \mathfrak{G} \leq \nu^q(G)$ and write $[g, h^q]$ in place of $g \otimes h$, for all $g, h \in G$. Following [10] we write $\Delta^q(G)$ for the subgroup $\langle [g, g^q] | g \in G \rangle \leq \Upsilon^q(G)$, which by Lemma 2.1 (vii) is a central subgroup of $\nu^q(G)$. We write $\tau^q(G)$ for the factor group $\nu^q(G)/\Delta^q(G)$. The subgroup $\Upsilon^q(G)/\Delta^q(G)$ of $\tau^q(G)$ is isomorphic to the q -exterior square $G \wedge^q G$. In order to avoid any confusion we usually write $[G, G^q]_{\tau(G)}$ to identify the q -exterior square $G \wedge^q G$ with the image of $[G, G^q]$ in $\tau^q(G)$. We shall eventually write T to denote the subgroup $[G, G^q]$ of $\nu^q(G)$ in order to distinguish it from the nonabelian tensor square $G \otimes G \cong [G, G^q] \leq \nu(G)$ in the case $q = 0$.

The material presented here incorporates part of the doctoral thesis [25] of the second named author, written under the supervision of the first.

2. SOME STRUCTURAL RESULTS

In this section we extend results found in [5] and [24] related to the non-abelian tensor square, from $G \otimes G$ to $G \otimes^q G$, $q \geq 0$. We begin by including some previous, technical results for future references.

The following basic properties are consequences of the defining relations of $\nu^q(G)$.

LEMMA 2.1. [10, Lemma 2.4] *Suppose that $q \geq 0$. The following relations hold in $\nu^q(G)$, for all $g, h, x, y \in G$.*

- (i) $[g, h^q]^{[x, y^q]} = [g, h^q]^{[x, y]}$;
- (ii) $[g, h^q, x^q] = [g, h, x^q] = [g, h^q, x] = [g^q, h, x^q] = [g^q, h^q, x] = [g^q, h, x]$;
- (iii) *If $h \in G'$ (or if $g \in G'$) then $[g, h^q][h, g^q] = 1$;*
- (iv) $[\hat{x}, [g, h^q]] = [\hat{x}, [g, h]]$;
- (v) $(\hat{x})^g = \hat{x}[x^q, g^q]$;

- (vi) If $[g, h] = 1$ then $[g, h^\varphi]$ and $[h, g^\varphi]$ are central elements of $\nu^q(G)$, of the same finite order dividing q . If in addition g, h are torsion elements of orders $o(g), o(h)$, respectively, then the order of $[g, h^\varphi]$ divides the $\gcd(q, o(g), o(h))$.
- (vii) $[g, g^\varphi]$ is central in $\nu^q(G)$, for all $g \in G$;
- (viii) $[g, h^\varphi][h, g^\varphi]$ is central in $\nu^q(G)$;
- (ix) $[g, g^\varphi] = 1$, for all $g \in G'$;
- (x) If $[x, g] = 1 = [x, h]$, then $[g, h, x^\varphi] = 1 = [[g, h]^\varphi, x]$.

COROLLARY 2.2. *Let G be any group and let g, h be arbitrary elements in G . Then*

- (i) $[G', G^\varphi] = [G, G'^\varphi]$;
- (ii) $[G', Z(G)^\varphi] = 1$;
- (iii) If $gG' = hG'$ then $[g, g^\varphi] = [h, h^\varphi]$;
- (iv) If $o'(x)$ denotes the order of a coset $xG' \in G/G'$, then $[g, h^\varphi][h, g^\varphi]$ has order dividing the $\gcd(q, o'(g), o'(h))$;
- (v) The order of $[h, h^\varphi]$ divides the $\gcd(q, o'(h)^2, 2o'(h))$.

Proof. Part (i) follows directly from Lemma 2.1 (iii) (see also [5, Corollary 1.2 (iii)]). As for part (ii), see [23, Proposition 2.7 (i)]. The remaining parts are appropriate adaptations of [24, Lemma 3.1 (v)], using (7) and Lemma 2.1 (vi). \square

For our purposes we establish the following proposition, which may have its own interest.

PROPOSITION 2.3. *Let G be a nilpotent group of class 2. Then the following hold in $\nu^q(G)$:*

- (i) \mathfrak{G} centralizes $[G, G^\varphi]$;
- (ii) $[G', G^\varphi] (= [G, G'^\varphi])$ is a central subgroup of $\nu^q(G)$;
- (iii) $\Upsilon^q(G) (\cong G \otimes^q G)$ is nilpotent of class at most 2.

Proof. (i) follows straightforward from 2.1 (iv) and relation (2), once G has nilpotency class 2.

(ii). For all $g, h \in G$ and $c \in G'$ we have:

$$\begin{aligned}
 [c, g^\varphi]^h &= [c, g^\varphi][c, g^\varphi, h] \\
 &= [c, g^\varphi][c, g, h^\varphi] && \text{(by Lemma 2.1, (i))} \\
 &= [c, g^\varphi] && \text{(since } [c, g] = 1, \text{ as } G' \leq Z(G)) \\
 &= [c, g^\varphi]^{h^\varphi} && \text{(by definition of } \nu^q(G)).
 \end{aligned}$$

In addition, for all $\widehat{k} \in \widehat{\mathcal{G}}$, by Lemma 2.1 (iv) and relations (2) we have that $\widehat{k}^{[c, g^\varphi]} = \widehat{k}^{[c, g]} = \widehat{k}$, since $[c, g] = 1$. This proves part (ii) (using the definition of $\nu^q(G)$), because $[G', G^\varphi]$ is generated by all those $[c, g^\varphi]$ above.

(iii). That $\Upsilon^q(G)$ is nilpotent and has nilpotency class at most 3 follows from [10, Proposition 2.7, (i)]. Now, $\Upsilon^q(G) = [G, G^\varphi]\mathfrak{G}$ and thus, once \mathfrak{G} centralizes $[G, G^\varphi]$, we have

$$(\Upsilon^q(G))' = [[G, G^\varphi]\mathfrak{G}, [G, G^\varphi]\mathfrak{G}] \leq [G, G^\varphi]'[\mathfrak{G}, \mathfrak{G}].$$

Induction arguments can be used, together with Lemma 2.1 (i), (ii), (iv) and (v) and defining relations (6) – (7) to get:

- (a) $[G, G^\varphi]' = [G', (G')^\varphi]$ (see also [5, Proposition 1.3 (i)] or [23, Theorem 3.3]);
- (b) $[\mathfrak{G}, \mathfrak{G}] \leq \langle \widehat{\mathcal{G}}' \rangle [G', G^\varphi] \leq [G', G^\varphi]$.

Consequently, $(\Upsilon^q(G))' \leq [G', G^\varphi]$, which by part (ii) is central in $\nu^q(G)$. This completes the proof. \square

For a finitely generated abelian group A , its q-tensor square $\Upsilon^q(A)$ can be computed by repeated applications of the following two results from [10].

LEMMA 2.4. [10, Corollary 2.16] *Let $G = N \times H$ be a direct product and set $\overline{N} = N/N'N^q$, $\overline{H} = H/H'H^q$. Then*

- (i) $\Upsilon^q(G) = \Upsilon^q(N) \times [N, H^\varphi][H, N^\varphi] \times \Upsilon^q(H)$;
- (ii) $[N, H^\varphi] \cong (\overline{N} \otimes_{\mathbb{Z}_q} \overline{H}) \cong [H, N^\varphi]$.

LEMMA 2.5. [10, Theorem 3.1] *Let C_n (resp. C_∞) be the cyclic group of order n (resp. ∞), q a non-negative integer and $d = \gcd(n, q)$. Then*

$$C_\infty \otimes^q C_\infty \cong C_\infty \times C_q;$$

$$C_n \otimes^q C_n \cong \begin{cases} C_n \times C_d, & \text{if } d \text{ is odd;} \\ C_n \times C_d, & \text{if } d \text{ is even and either } 4|n \text{ or } 4|q; \\ C_{2n} \times C_{d/2}, & \text{otherwise.} \end{cases}$$

Thus, if $A = \prod_{i=1}^r C_i$ is a direct product of the cyclic groups C_i , $i = 1, \dots, r$, where $C_i = \langle x_i \rangle$, then

$$\Upsilon^q(A) = \prod_{i=1}^r \Upsilon^q(C_i) \times \prod_{1 \leq i < j \leq r} [C_i, C_j^\varphi][C_j, C_i^\varphi].$$

Here we have $\Upsilon^q(C_i) = \langle [x_i, x_i^\varphi], \widehat{x_i} \rangle$ and $[C_i, C_j^\varphi][C_j, C_i^\varphi] = \langle [x_i, x_j^\varphi][x_j, x_i^\varphi], [x_i, x_j^\varphi] \rangle$. Since $\Delta^q(A) = \langle [a, a^\varphi] \mid a \in A \rangle$, we observe, like in [24, Proposition 3.3], that $\Delta^q(A) = \langle [x_i, x_i^\varphi], [x_j, x_k^\varphi][x_k, x_j^\varphi] \mid 1 \leq i \leq r, 1 \leq j < k \leq r \rangle$ and thus it does not depend on the particular set $X = \{x_1, \dots, x_r\}$ of generators of A . Consequently, we can write

$$\Upsilon^q(A) = \Delta^q(A)E_X^q(A),$$

where $E_X^q(A) = \langle \widehat{x_i}, [x_j, x_k^\varphi] \mid 1 \leq i \leq r, 1 \leq j < k \leq r \rangle$.

REMARK 2.6. *If x and y are commuting elements in any group G then by relations (5)–(7) and Lemma 2.1 (vi) we get*

$$\widehat{xy} = \widehat{x}\widehat{y}[x, y^\varphi]^{-\binom{q}{2}} = \widehat{x}\widehat{y}[y, x^\varphi]^{-\binom{q}{2}} = \widehat{yx},$$

and hence $[x, y^\varphi]^{-\binom{q}{2}} = [y, x^\varphi]^{-\binom{q}{2}}$. In particular, if $q = 2$ then $[x, y^\varphi] = [y, x^\varphi]$. This means for instance that in the decomposition of $\Upsilon^q(A)$ found above, the groups $[C_i, C_j^\varphi]$ and $[C_j, C_i^\varphi]$ are not necessarily independent. Moreover, the identity $(\widehat{x^n}) = (\widehat{x})^n[x, x^\varphi]^{-\binom{n}{2}\binom{q}{2}}$ shows that the subgroups $\langle \widehat{x_i} \rangle$ and $\langle [x_i, x_i^\varphi] \rangle$ of $\Upsilon^q(C_i)$ may have non trivial intersection. Consequently, unlike the case $q = 0$, the subgroup $E_X^q(A)$ is not necessarily a complement of $\Delta^q(A)$ (see also [5, Section 2]).

Now let G be any group and write $G^{ab} = G/G'$. The natural projection $G \twoheadrightarrow G^{ab}$ induces an epimorphism $\pi : \nu^q(G) \rightarrow \nu^q(G^{ab})$. We denote by π_0 the restriction of π to $\Upsilon^q(G)$. By [10, Lemma 2.14 (iii)] we have that $\text{Ker}(\pi_0) = [G', G^\varphi][G, G'^\varphi]\langle \widehat{\mathcal{G}'} \rangle$, which reduces to $[G', G^\varphi]\langle \widehat{\mathcal{G}'} \rangle$, by force of Corollary 2.2 (i). In addition, using relations (5) and (7), an induction argument as in the proof of the Proposition 2.3 (ii) shows that $\langle \widehat{\mathcal{G}'} \rangle \leq [G', G^\varphi]$ and, consequently, $\text{Ker}(\pi_0) = [G', G^\varphi]$.

The next Lemma extends [5, Lemma 2.1] to all $q \geq 0$ (see also [24, Proposition 3.3]). We shall omit the proof.

LEMMA 2.7. *Let q be a non negative integer and G be a group such that G^{ab} is finitely generated. Assume that G^{ab} is a direct product of the cyclic groups $C_i = \langle x_i G' \rangle$, for $i = 1, \dots, r$ and set*

$$E^q(G) = \langle \widehat{x}_i, [x_j, x_k^\varphi] \mid 1 \leq i \leq r, 1 \leq j < k \leq r \text{ range } [G', G^\varphi] \rangle.$$

Then,

- (i) $\Delta^q(G) = \langle [x_i, x_i^\varphi], [x_j, x_k^\varphi][x_k, x_j^\varphi] \mid 1 \leq i \leq r, 1 \leq j < k \leq r \rangle$;
- (ii) $\Upsilon^q(G) = \Delta^q(G)E^q(G)$.

With the above notation, let π_1 denote the restriction of π_0 to $\Delta^q(G)$, $\pi_1 : \Delta^q(G) \twoheadrightarrow \Delta^q(G^{ab})$, and let $N = \text{Ker}(\pi_1)$. Therefore, $N = \Delta^q(G) \cap [G', G^\varphi] (= \Delta^q(G) \cap E^q(G))$, a central subgroup of $\Upsilon^q(G)$.

Our next theorem generalizes, to all $q \geq 0$, Proposition 2.2 in [5], which in turn improves Proposition 3.3 in [24].

THEOREM 2.8. *Let $q \geq 0$ and assume that G^{ab} is finitely generated. Then, with the notation of Lemma 2.7, the following hold:*

- (i) $\Upsilon^q(G)/N \cong \Delta^q(G^{ab}) \times (G \wedge^q G)$;
- (ii) *If $q \geq 1$ and q is odd, then $N = 1$ and thus $\Delta^q(G) \cong \Delta^q(G^{ab})$ and $\Upsilon^q(G) \cong \Delta^q(G^{ab}) \times (G \wedge^q G)$;*
- (iii) *For $q = 0$ or $q \geq 2$ and q even, if G^{ab} has no element of order two or if G' has a complement in G , then also $N = 1$, $\Delta^q(G) \cong \Delta^q(G^{ab})$ and $\Upsilon^q(G) \cong \Delta^q(G^{ab}) \times (G \wedge^q G)$;*
- (iv) *For $q \geq 2$ and q even, if G^{ab} has no element of order 2, then $\Delta^q(G)$ is a homocyclic abelian group of exponent q , of rank $\binom{t+1}{2}$;*
- (v) *If G^{ab} is free abelian of rank t , then the conclusion of the previous item holds for all $q \geq 1$, while $\Delta^q(G)$ is free abelian of rank $\binom{t+1}{2}$ if $q = 0$.*

Proof. (i): By Lemma 2.7 (ii) we have

$$\frac{\Upsilon^q(G)}{N} = \frac{\Delta^q(G)E^q(G)}{N} \cong \frac{\Delta^q(G)}{N} \times \frac{E^q(G)}{N}.$$

Now, $\Delta^q(G)/N \cong \Delta^q(G^{ab})$ and

$$E^q(G)/N = E^q(G)/(\Delta^q(G) \cap E^q(G)) \cong \Upsilon^q(G)/\Delta^q(G) \cong G \wedge^q G.$$

This proves (i).

(ii), (iii), (iv) and (v): Suppose that the torsion subgroup of G^{ab} is the direct product of the cyclic groups $\langle x_i G' \rangle$ of order n_i , $1 \leq i \leq s$, and let the free part of G^{ab} be the direct product of the cyclic groups $\langle y_j G' \rangle$, $1 \leq j \leq t$. Thus, $o'(x_i) = n_i$ and $o'(y_j) = \infty$. Set $X := \{x_i \mid 1 \leq i \leq s\}$ and $Y := \{y_j \mid 1 \leq j \leq t\}$. Then G is generated by $X \cup Y \cup G'$. Using Lemma 2.1 and Corollary 2.2 (see also [24, Proposition 3.3 and Remark 5]) we find that $\Delta^q(G)$ is generated by the set $\Delta_X \cup \Delta_Y \cup \Delta_{XY}$, where

$$\begin{aligned}\Delta_X &= \{[x_i, x_i^\varphi], [x_j, x_k^\varphi][x_k, x_j^\varphi] \mid 1 \leq i \leq s, 1 \leq j < k \leq s\}, \\ \Delta_Y &= \{[y_j, y_j^\varphi], [y_k, y_l^\varphi][y_l, y_k^\varphi] \mid 1 \leq j \leq t, 1 \leq k < l \leq t\}, \\ \Delta_{XY} &= \{[x_i, y_j^\varphi][y_j, x_i^\varphi] \mid 1 \leq i \leq s, 1 \leq j \leq t\}.\end{aligned}$$

Set $n_{ik} = \gcd(n_i, n_k)$. Parts (iv) and (v) of Corollary 2.2 give $([x_i, x_k^\varphi][x_k, x_i^\varphi])^{n_{ik}} = 1$ and $([x_i, y_j^\varphi][y_j, x_i^\varphi])^{n_i} = 1$, while $[x_i, x_i^\varphi]^{n_i} \in \text{Ker}(\pi_0)$, $\forall i, k = 1, \dots, s, i < k, \forall j = 1, \dots, t$. Actually, $\text{Ker}(\pi_0)$ is generated by the set $\{[x_i, x_i^\varphi]^{n_i} \mid 1 \leq i \leq s\}$, by [24, Proposition 3.5]. Again by Corollary 2.2 (v), we get that if $n_i (= o'(x_i))$ is odd, then $[x_i, x_i^\varphi]^{n_i} = 1$, while $[x_i, x_i^\varphi]^{2n_i} = 1$ if n_i is even. Consequently, $N = \text{Ker}(\pi_0)$ is an elementary abelian 2-group of rank at most $r_2(G^{ab})$, the 2-rank of G^{ab} (see also [24, Corollary 3.6]). On the other hand, we should take into account that q is involved in the upper bound found in Corollary 2.2 (v). Thus, if $q \geq 1$ and q is odd, then $\gcd(q, 2n_i) = \gcd(q, n_i) \mid n_i$ and hence $[x_i, x_i^\varphi]^{n_i} = 1$, for all $i = 1, \dots, s$. Therefore $N = 1$ in this case, proving part (ii). It should be also clear that $N = 1$ if $r_2(G^{ab}) = 0$. This proves (iii) in the case where G^{ab} has no element of order 2. Now, if G' has a complement C in G , then every $g \in G$ can be written as $g = xh$ with $x \in C$ and $h \in G'$. Corollary 2.2 (iii) says that $[g, g^\varphi] = [x, x^\varphi]$ and thus $\Delta^q(G) = \langle [x, x^\varphi] \mid x \in C \rangle = \Delta^q(G^{ab})$. This completes the proof of part (iii) (see also [5, Proposition 2.2]). Finally, we observe that $[x_i, x_i^\varphi] = 1 = [x_i, y_j^\varphi][y_j, x_i^\varphi]$ in the case where $r_2(G^{ab}) = 0$ and $q \geq 2$, q even. Here we have $\Delta^q(G) = \langle \Delta_Y \rangle$ and $[y_j, y_j^\varphi]^q = 1 = ([y_k, y_l^\varphi][y_l, y_k^\varphi])^q, \forall j, k, l = 1, \dots, t, k < l$. That each of these $\binom{t+1}{2}$ generators has order q follows immediately from Lemma 2.5, where we found $C_\infty \otimes^q C_\infty \cong C_\infty \times C_q$. Part (v) follows by an analogous argument, as in part (iv); the last assertion can be also found in [24, Corollary 3.6]. The proof is complete. \square

We state the next Lemma for easy of reference, which in a certain sense extends ideas found in [19] for the case $q = 0$. A proof is given in [8] for $q = 0$ (see also [5, Proposition 3.2] for an alternative proof for this case) and in [16] for $q \geq 1$.

LEMMA 2.9. *Let F/R be a free presentation of a group G . Then*

$$G \wedge^q G \cong F' F^q / [R, F] R^q.$$

Notice that there is a map

$$\rho : \Upsilon^q(G) \longrightarrow G, g \longmapsto g, g^\varphi \longmapsto g \text{ and } \widehat{k} \longmapsto k^q.$$

Let $\rho' = \rho|_{\Upsilon^q(G)} : \Upsilon^q(G) \longrightarrow G, [g_1, g_2^\varphi] \longmapsto [g_1, g_2], \widehat{k} \longmapsto k^q$.

Following [10] we write $\theta^q(G) = \text{Ker}(\rho)$ and $\mu^q(G) = \text{Ker}(\rho') = \Upsilon^q(G) \cap \theta^q(G)$. It follows that $\Upsilon^q(G)/\mu^q(G) \cong G' G^q$. If $G = F/R$ is a free presentation of G , then

$$(10) \quad H^2(G, \mathbb{Z}_q) \cong R \cap F' F^q / R^q [R, F] = (G \wedge^q G) \cap M^q(G),$$

where $M^q(G) = R/R^q[R, F]$ is the q -multiplier of G . From this we obtain (see for instance [10, Theorem 2.12]):

$$(11) \quad \mu^q(G)/\Delta^q(G) \cong H^2(G, \mathbb{Z}_q),$$

for all $q \geq 0$.

COROLLARY 2.10. *Let F_n be the free group of rank n . Then*

(i) *For $q \geq 1$,*

$$F_n \otimes^q F_n \cong C_q^{\binom{n+1}{2}} \times (F_n)'(F_n)^q.$$

(ii) ([9, Proposition 6]) *For $q = 0$,*

$$F_n \otimes F_n \cong C_\infty^{\binom{n+1}{2}} \times (F_n)'.$$

Proof. Since F_n^{ab} is free abelian of rank n , by Theorem 2.8 (ii), (iii) and (v), we have:

$$\Upsilon^q(F_n) \cong \Delta^q(F_n^{ab}) \times (F_n \wedge^q F_n).$$

(i): If $q \geq 1$ then $\Delta^q(F_n^{ab}) \cong C_q^{\binom{n+1}{2}}$ and, by Lemma 2.9 (i) with $G = F_n$ and $R = 1$,

$$F_n \wedge^q F_n \cong (F_n)'(F_n)^q.$$

This proves (i).

(ii): If $q = 0$ then $\Delta^q(F_n^{ab}) \cong C_\infty^{\binom{n+1}{2}}$ and, again by the previous Lemma,

$$F_n \wedge F_n \cong (F_n)'.$$

This completes the proof. \square

COROLLARY 2.11. *Let $\mathcal{N}_{n,c} = F_n/\gamma_{c+1}(F_n)$ be the free nilpotent group of class $c \geq 1$ and rank $n > 1$. Then*

(i) *For $q \geq 1$,*

$$\mathcal{N}_{n,c} \otimes^q \mathcal{N}_{n,c} \cong C_q^{\binom{n+1}{2}} \times \frac{(F_n)'(F_n)^q}{\gamma_{c+1}(F_n)^q \gamma_{c+2}(F_n)}.$$

(ii) ([6, Corollary 1.7]) *For $q = 0$,*

$$\mathcal{N}_{n,c} \otimes \mathcal{N}_{n,c} \cong C_\infty^{\binom{n+1}{2}} \times (\mathcal{N}_{n,c+1})'.$$

Proof. (i) and (ii) follow by similar arguments as above, taking into account that here we have $R = \gamma_{c+1}(F_n)$ and thus $[R, F]$, as in Lemma 2.9, is precisely $\gamma_{c+2}(F_n)$. \square

3. Q-TENSOR SQUARES OF NILPOTENT GROUPS OF CLASS 2

In this section we restrict our considerations to finitely generated nilpotent groups G of class two. We begin with a general result concerning polycyclic groups found in [10]; this generalizes to all $q \geq 0$ a result due to Blyth and Morse in [4] for $q = 0$, which in turn extends to all polycyclic groups a similar result for finite solvable groups found in [24].

LEMMA 3.1. ([10, Corollary 3.6]) *Let G be a polycyclic group with a polycyclic generating sequence $\mathbf{pgs}(G) = (\mathbf{a}_1, \dots, \mathbf{a}_n)$. Then*

(i) $[G, G^\varphi]$, a subgroup of $\nu^q(G)$, $q \geq 0$, is generated by

$$[G, G^\varphi] = \left\langle [\mathbf{a}_i, \mathbf{a}_i^\varphi], [\mathbf{a}_i, \mathbf{a}_j^\varphi][\mathbf{a}_j^\varphi, \mathbf{a}_i], [\mathbf{a}_i^\alpha, (\mathbf{a}_j^\varphi)^\beta], \text{ for } 1 \leq i < j \leq n, 1 \leq k \leq n \right\rangle,$$

(ii) $\Upsilon^q(G)$, a subgroup of $\nu^q(G)$, $q \geq 1$, is generated by

$$\Upsilon^q(G) = \left\langle [\mathbf{a}_i, \mathbf{a}_i^\varphi], [\mathbf{a}_i, \mathbf{a}_j^\varphi][\mathbf{a}_j^\varphi, \mathbf{a}_i], [\mathbf{a}_i^\alpha, (\mathbf{a}_j^\varphi)^\beta], \widehat{(\mathbf{a}_k)}, \text{ for } 1 \leq i < j \leq n, 1 \leq k \leq n \right\rangle,$$

$$\text{where } \alpha = \begin{cases} 1 & \text{if } o(\mathbf{a}_i) < \infty \\ \pm 1 & \text{if } o(\mathbf{a}_i) = \infty \end{cases} \text{ and } \beta = \begin{cases} 1 & \text{if } o(\mathbf{a}_j) < \infty \\ \pm 1 & \text{if } o(\mathbf{a}_j) = \infty. \end{cases}$$

(iii) $\Delta^q(G)$ is generated by the set $\{[\mathbf{a}_i, \mathbf{a}_i^\varphi], [\mathbf{a}_i, \mathbf{a}_j^\varphi][\mathbf{a}_j^\varphi, \mathbf{a}_i], \text{ for } 1 \leq i < j \leq n\}$.

Now let G be a finitely generated nilpotent group of class two and assume that G is generated by g_1, g_2, \dots, g_n . Thus, any element $g \in G$ can be written as

$$(12) \quad g = \prod_{i=1}^n g_i^{m_i} \prod_{1 \leq j < k \leq n} [g_j, g_k]^{l_{jk}},$$

where the exponents m_i and l_{jk} are integers. Consequently, G has the following polycyclic generating set

$$(13) \quad \{g_i, 1 \leq i \leq n\} \cup \{[g_j, g_k], 1 \leq j < k \leq n\}.$$

The following theorem extends a result of Bacon in [2, Theorem 3.1] (see also [3]) for all $q \geq 0$. We provide a proof for the case $q = 0$ using the commutator approach; the general case follows straightforward from this case and Lemma 3.1 (ii), but we shall prove it in this case too, for the sake of completeness.

THEOREM 3.2. *Let G be a nilpotent group of class two with $d(G) = n$, then*

- (i) ([2, Theorem 3.1]) $d([G, G^\varphi]) \leq \frac{n(n^2+3n-1)}{3}$;
- (ii) $d(G \otimes^q G) \leq \frac{n(n^2+3n+2)}{3}$, for all $q \geq 0$;
- (iii) In particular, if G has finite exponent $e(G)$ and $\gcd(q, e(G)) = 1$, then $d(G \otimes^q G) \leq n^2$.

Proof. On assuming that G is generated by g_1, \dots, g_n then we obtain the polycyclic generating set given by (13). Thus, by Lemma 3.1 (i) we have that $\Upsilon(G) = [G, G^\varphi]$ is generated by the following set of elements:

$$\begin{aligned} & \{[g_i^\alpha, (g_j^\varphi)^\beta] : 1 \leq i, j \leq n\} \cup \\ & \{[g_i^\alpha, ([g_j, g_k]^\varphi)^\beta] : 1 \leq i \leq n, 1 \leq j < k \leq n\} \cup \\ & \{[[g_j, g_k]^\beta, (g_i^\varphi)^\alpha] : 1 \leq i \leq n, 1 \leq j < k \leq n\} \cup \\ & \{[[g_r, g_s]^\alpha, ([g_t, g_u]^\varphi)^\beta] : 1 \leq r < s \leq n, 1 \leq t < u \leq n\}, \quad \text{where } \alpha, \beta \in \{-1, 1\}. \end{aligned}$$

Now by Lemma 2.1, parts (ii), (iii), (ix), (x), and the fact that G has class 2, we can further reduce the above set to obtain

$$(14) \quad \{[g_i, g_j^\varphi] : 1 \leq i, j \leq n\} \cup \{[g_i, [g_j, g_k]^\varphi] : 1 \leq i \leq n, 1 \leq j < k \leq n\}.$$

This new set has n^2 generators of the form $[g_i, g_j^\varphi]$ and $n(n-1)$ generators of the form $[g_i, [g_j, g_k]^\varphi]$. It remains to count the generators of the form $[g_i, [g_j, g_k]^\varphi]$, when i, j, k are all distincts and $j < k$. Now by [23, Corollary 3.2] $\nu(G)$ has nilpotency class at most 3 and thus, again

$$[g_j^\varphi, g_k^\varphi, g_i][g_k^\varphi, g_i^\varphi, g_j][g_i, g_j, g_k^\varphi] = 1.$$

It then follows that $[g_i, [g_j, g_k]^\varphi] = [g_j, [g_i, g_k]^\varphi][g_k, [g_j, g_i]^\varphi]$. Therefore,

$$d([G, G^\varphi]) \leq \frac{1}{3}n(n^2 + 3n - 1).$$

This proves part (i).

(ii): Part (i) also proves (ii) in case $q = 0$, giving us the better bound for $d(G \otimes G)$. Thus, we shall assume $q \geq 1$. Since $\Upsilon^q(G) = [G, G^\varphi]\mathfrak{G}$ it suffices to control the number of generators of both $[G, G^\varphi]$ and \mathfrak{G} . We already know by part (i) that $[G, G^\varphi]$ is generated by the set $\{[g_i, g_j^\varphi] : 1 \leq i, j \leq n\} \cup \{[g_i, [g_j, g_k]^\varphi] : 1 \leq i \leq n, 1 \leq j < k \leq n\}$. Now, by definition the subgroup \mathfrak{G} is generated by $\widehat{\mathcal{G}} = \{\widehat{g}, g \in G\}$. By the defining relations (5) of $\nu^q(G)$ we have $\widehat{g}\widehat{h} = \widehat{g}(\prod_{i=1}^{q-1}[g, (h^\varphi)^{-i}]^{g^{q-1-i}})\widehat{h}$, for all $g, h \in G$, and hence $\widehat{g}\widehat{h} \equiv \widehat{g}\widehat{h} \pmod{[G, G^\varphi]}$, for all $\widehat{g}, \widehat{h} \in \widehat{\mathcal{G}}$. Since every element $g \in G$ has a unique expression in the form (12) and given the fact that, for every commutator $[g_j, g_k] \in G'$, $\widehat{[g_j, g_k]} = [g_j, g_k^\varphi]^q \in [G, G^\varphi]$ (by relations (7)), we see in later stage that \mathfrak{G} is generated, modulo $[G, G^\varphi]$, by the n elements $\widehat{g}_1, \dots, \widehat{g}_n$. Therefore, we conclude that $d(\Upsilon^q(G)) \leq \frac{1}{3}(n^3 + 3n^2 + 2n)$. This proves part (ii).

(iii): If in particular G has finite exponent $e(G)$ and $\gcd(q, e(G)) = 1$, then we see by Lemma 2.1 (vi) that all generators of the forms $[g_i, g_i^\varphi]$ and $[g_i, [g_j, g_k]^\varphi]$ are trivial. Consequently, $d(G \otimes^q G) \leq n^2$ in this case. The proof is complete. \square

In [1] Aoughazi computed the nonabelian tensor square of the Heisenberg group $\mathcal{H} = F_2/\gamma_3(F_2)$, where F_2 denotes the free group of rank 2. There, it is found that $\mathcal{H} \otimes \mathcal{H} \cong \mathbb{Z}^6$, thus showing that the bound in Theorem 3.2 is sharp. Later, Bacon in [2, Theorem 3.2] computed $\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2}$ for all $n \geq 2$, to show that the bound is also reached for the free n -generator nilpotent group of class 2, $n > 1$: $\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2}$ is a free abelian group of rank $\frac{1}{3}n(n^2 + 3n - 1)$.

It is not difficult to extend these results to the q -tensor square $\Upsilon^q(\mathcal{N}_{n,2})$, $q \geq 1$, to show that the bound in Theorem 3.2 (ii) is also attained. In fact, the next proposition is but a specialization of Corollary 2.11. We write $M(G)$ to denote the Schur multiplier $H_2(G, \mathbb{Z})$ of G .

PROPOSITION 3.3. *Let $\mathcal{N}_{n,2}$ be the free nilpotent group of rank $n > 1$ and class 2, $\mathcal{N}_{n,2} = F_n/\gamma_3(F_n)$. Then,*

- (i) ([2, Theorem 3.2]) $\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2}$ is free abelian of rank $\frac{1}{3}n(n^2 + 3n - 1)$. More precisely,

$$\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2} \cong \Delta(F_n^{ab}) \times M(\mathcal{N}_{n,2}) \times \mathcal{N}'_{n,2}.$$

- (ii) $\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2} \cong (C_q)^{\binom{n+1}{2} + M_n(3)} \times \mathcal{N}'_{n,2} \mathcal{N}_{n,2}^q$, where $M_n(3) = \frac{1}{3}(n^3 - n)$ is the q -rank of $\gamma_3(\mathcal{N}_{n,2})/\gamma_3(\mathcal{N}_{n,2})^q\gamma_4(\mathcal{N}_{n,2})$, according to the Witt's formula.

Consequently, for $q > 1$

$$d(\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2}) = \frac{1}{3}(n^3 + 3n^2 + 2n).$$

Proof. As in the proof of Corollary 2.11, using Lemma 2.9 we can write

$$G \wedge^q G \cong F'_n F_n^q / [R, F_n] R^q.$$

In view of (10),

$$H^2(G, \mathbb{Z}_q) \cong R \cap F'_n F_n^q / R^q [R, F_n].$$

Taking into account that $R = \gamma_3(F_n) \leq F'_n F_n^q$ we have the exact sequence

$$(15) \quad 1 \rightarrow \frac{\gamma_3(F_n)}{\gamma_4(F_n)\gamma_3(F_n)^q} \rightarrow \frac{F'_n F_n^q}{\gamma_4(F_n)\gamma_3(F_n)^q} \rightarrow \frac{F'_n F_n^q}{\gamma_3(F_n)} \rightarrow 1.$$

Here we find that $H_2(\mathcal{N}_{n,2}, \mathbb{Z}_q) \cong \gamma_3(F_n) / \gamma_4(F_n) \gamma_3(F_n)^q \cong \mathbb{Z}_q^{M_n(3)}$, for all $q \geq 0$, where, by the Witt's formula for the rank of $\gamma_r(F_n) / \gamma_{r+1}(F_n)$,

$$M_n(r) = \frac{1}{r} \sum_{d|r} \mu(d) n^{\frac{r}{d}},$$

where μ denotes the Möbius function.

(i): If $q = 0$ then the exact sequence (15) splits and thus we have, by Corollary 2.11,

$$\mathcal{N}_{n,c} \otimes \mathcal{N}_{n,c} \cong C_\infty^{\binom{n+1}{2}} \times \gamma_3(F_n) / \gamma_4(F_n) \times \gamma_2(F_n) / \gamma_3(F_n) \cong C_\infty^{\binom{n+1}{2}} \times C_\infty^{\binom{n}{2}} \times C_\infty^{\frac{1}{3}(n^3-n)}.$$

This is the result of Bacon, that $d(\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2}) = \frac{1}{3}n(n^2 + n - 1)$.

(ii): If $q > 1$ then we see by the generators of $\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2}$ found in Theorem 3.2 that the image by ρ' of the subgroup $\langle \widehat{g}_i, [g_j, g_k^q] \mid 1 \leq i \leq n, 1 \leq k < j \leq n \rangle$ is the subgroup $\mathcal{N}'_{n,2} \mathcal{N}_{n,2}^q$ of $\mathcal{N}_{n,2}$, while the subgroup $\langle [gj, g_i, g_k^q] \rangle$, where $[gj, g_i, g_k]$ is a basic commutator of $\gamma_3(F_n) / \gamma_4(F_n)$, is isomorphic to $\gamma_3(F_n) / \gamma_4(F_n) \gamma_3(F_n)^q \cong H_2(\mathcal{N}_{n,2}, \mathbb{Z}_q)$, a homocyclic abelian group of exponent q and q -rank $M_n(3) = \frac{1}{3}(n^3 - n)$. Consequently, also in this case we get that the sequence (15) splits and we then find that

$$\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2} \cong C_q^{\binom{n+1}{2} + M_n(3)} \times \mathcal{N}'_{n,2} \mathcal{N}_{n,2}^q.$$

Therefore, $d(\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2}) = \frac{1}{3}n(n^2 + 3n + 2)$, thus showing that the upper bound given in Theorem 3.2 (ii) is also achieved when $q > 1$. \square

REFERENCES

- [1] Aboughazi, R., *Produit Tensoriel du Groupe D'Heisenberg*, Bull. Soc. math. France, **115** (1987), 95–106.
- [2] Bacon, M., *On the non-abelian Tensor Square of a Nilpotent Group of Class Two*, Glasgow Math. J., **3** (1994), 291–295.
- [3] Bacon, M. and L.-C. Kappe *The nonabelian tensor square of a 2-generator p-group of class 2*. Arch. Math. (Basel) **61** (1993), 508–516.
- [4] Blyth, R. D., Morse, R. F., *Computing the nonabelian tensor square of polycyclic groups*, J. Algebra **321** (2009), 2139–2148.
- [5] Blyth, R. D., Fumagalli, F., Morigi, M., *Some structural results on the non-abelian tensor square of groups*, J. Group Theory, **13**, No.1 (2010), 83–94.

- [6] Blyth, R. D., Moravec, P., Morse, R. F. *On the nonabelian tensor squares of free nilpotent groups of finite rank*. In *Computational Group Theory and the Theory of Groups*, Contemporary Mathematics 470 (American Mathematical Society, 2008), pp. 27–44.
- [7] Brown, R., *q-perfect Groups and Universal q-central Extensions*, Publ. Mat., **34** (1990), 291–297.
- [8] Brown, R. and Loday, J.-L., *Van Kampen Theorems for Diagrams of Spaces*, Topology, **26** (1987), 311–335.
- [9] Brown, R., Johnson, D. L., Robertson, E. F., *Some computations of non-abelian tensor products of groups*, J. Algebra **111** (1987), 177–202, .
- [10] Bueno, T. P. and Rocco, N. R., *On the q-tensor square of a group*, J. Group Theory, **14** (2011), 785–805.
- [11] Conduché D., Rodríguez-Fernández, C., *Non-abelian Tensor and Exterior Products modulo q and Universal q-central Relative Extensions*, J. Pure Appl. Algebra, **78**, No.2 (1992), 139–160.
- [12] Eick, B., Nickel, W., *Computing the Schur multiplier and the nonabelian tensor square of a polycyclic group*, J. Algebra **320** No.2 (2008), 927–944.
- [13] Eick, B., Nickel, W., *POLYCYCLIC - Computation with polycyclic groups, A GAP 4 package*. In *The Gap Group, GAP—Groups, Algorithms, and Programming*, Version 4.5.7, 2014 (<http://www.gap-system.org>).
- [14] Ellis, G., *Tensor Product and q-crossed Modules*, J. London Math. Soc., **2** (51) No.2 (1995), 243–258.
- [15] Ellis, G., Leonard, F., *Computing Schur multipliers and tensor products of finite groups*, Proc. Royal Irish Acad., 95A (1995), 137–147.
- [16] Ellis, G., Rodríguez-Fernández, C., *An exterior product for the homology of groups with integral coefficients modulo p*, Cah. Top. Géom. Diff. Cat. 30, 339–343, (1989).
- [17] Gilbert, N. D., Higgins, P. J., *The non-abelian tensor product of groups and related constructions*, Glasgow Math. J. **31** (1989), 17–29.
- [18] McDermott, A., *The Nonabelian Tensor Product of Groups: Computations and Structural Results*, PhD Thesis, Nat. Univ. Ireland, Galloway, 1998.
- [19] Miller, C., *The Second Homology Group of a Group; Relations Among Commutators*, Proc. Amer. Math. Soc. **3** (1952), 588–595.
- [20] Nakaoka, I. N., *Non abelian tensor products of solvable groups*, J. Group Theory **3** (2000), 157–167.
- [21] Nakaoka, I. N. and Rocco, N. R., *A survey of non-abelian tensor products of groups and related constructions*, Bol. Soc. Paran. Mat. **30** 1 (2012), 77–89.
- [22] Robinson, Derek J. S., *A Course in the Theory of Groups*, second edition, Graduate Texts in Mathematics **80**, Springer-Verlag New York (1996).
- [23] Rocco, N. R., *On a Construction Related to the Non-Abelian Tensor Square of a Group*, Bol. Soc. Bras. Mat., **22**, No.1 (1991), 63–79.
- [24] Rocco, N. R., *A Presentation for a Crossed Embedding of Finite Solvable Groups*, Comm. in Algebra, **22**(6) (1994), 1975–1998.
- [25] Rodrigues, Eunice C. P., *Cotas Superiores para o Expoente e o número mínimo de geradores do Quadrado q-Tensorial de Grupos Nilpotentes*, Doctoral Thesis (in Portuguese), Universidade de Brasília, Brasília, DF, Brazil, 2011. Available at <http://repositorio.unb.br/handle/10482/8717>.

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